ON THE THEORY OF STABILITY OF FLOW OF VISCOUS FLUIDS IN CHANNELS

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The stability of the flow of a viscous incompressible fluid in a two-dimensional channel with parallel walls has been studied in detail in the case of symmetric velocity profiles in [1,2]. There are possible flow patterns, however, the velocity profiles of which are not symmetric. Such flow patterns arise, for example, in channels with permeable walls, through which the fluid moves with different velocities [3].

This paper analyses the stability of almost parallel flows with asymmetric velocity profiles both in two-dimensional and annular channels, using the method developed in [1]. The difficulty of the problem lies in the fact that the asymmetric velocity profile does not allow one to consider just symmetric or antisymmetric perturbations. In the analysis we use the results of mathematical papers on the behavior of asymptotic solutions of the Orr-Sommerfeld equation [4,5].

1. The problem of the stability of parallel flows reduces to the solution of the Orr-Sommerfeld equation for the amplitude of the stream function of the perturbation flow of ϕ

$$(D^2-\alpha^2)^2\varphi = i\alpha R\,[(w-c)(D^2-\alpha^2)-D^2w]\,\varphi \qquad \left(D^2=\frac{d^2}{dy^2}\right) \qquad (1.1)$$

Here y is the dimensionless coordinate measured along the normal to the wall of the channel, $\alpha = 2\pi H/\lambda$ is the wave number, λ is the wavelength of the perturbation, and H is the width of the channel; $R = UH/\nu$ is the Reynolds number of the stream, corresponding to the maximum velocity U in the given section of the channel ; ν is the kinematic viscosity of the fluid; $w = u/U$ is the dimensionless velocity profile; c is the dimensionless wave velocity, which we shall assume to be a real number, considering only neutrally stable oscillations.

Satisfying the boundary conditions

$$\varphi = \varphi' = 0 \quad \text{when } y = y_1 \text{ and } y = y_2 \qquad (1.2)$$

we obtain the equation for the eigenvalues

$$\begin{vmatrix} \varphi_{11} & \varphi_{21} & \varphi_{31} & \varphi_{41} \\ \varphi_{12} & \varphi_{22} & \varphi_{32} & \varphi_{42} \\ \varphi_{11}' & \varphi_{21}' & \varphi_{31}' & \varphi_{41}' \\ \varphi_{12}' & \varphi_{22}' & \varphi_{32}' & \varphi_{42}' \end{vmatrix} = 0 \qquad (1.3)$$

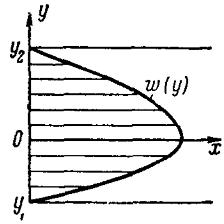

Fig. 1.

where $\phi_{kl} = \phi_k(y_l)$ $(k = 1, 2, 3, 4;\ l = 1, 2)$ are the values of the linearly independent solutions of Equation (1.1) at the points y_1 and y_2.

For large values of the parameter aR two solutions of Equation (1.1), which we shall denote by ϕ_1 and ϕ_2, may with sufficient accuracy be taken as the solutions of the degenerate equation

$$(w - c)(\varphi'' - \alpha^2\varphi) - w''\varphi = 0 \qquad (1.4)$$

obtained from (1.1) as $aR \to \infty$. The other two solutions, ϕ_3 and ϕ_4, essentially depend on the forces of viscosity, and their asymptotic expressions have the form

$$\varphi_3 = A(y)\, e^{-Y(y)}, \qquad \varphi_4 = B(y)\, e^{Y(y)} \qquad \left(Y(y) = \int_{y_{c1}}^{y} \sqrt{i\alpha R(w - c)}\, dy\right) \qquad (1.5)$$

where the functions $A(y)$ and $B(y)$ vary weakly in comparison with the exponential factors, whilst y_{cl} is the coordinate of the critical point, lying near y_1. Regarding the function $w(y)$ we shall assume that it increases monotonically (Fig. 1) from the point y_1 to $y = 0$, whilst thereafter it decreases monotonically to the point y_2; $w(y_1) = w(y_2) = 0$ and therefore the equation $w(y) = c$ has two real roots, y_{c1} and y_{c2}. Expressions (1.5) will be used to simplify the equation for the eigenvalues (1.3). We shall use the notation

$$P = \int_{y_1}^{y_2} \sqrt{i\alpha R(w - c)}\, dy, \qquad A_l = A(y_l), \qquad B_l = B(y_l) \quad (l = 1,2) \qquad (1.6)$$

Then

$$\frac{\varphi_{32}}{\varphi_{31}} = \frac{A_2}{A_1} e^{-P}, \qquad \frac{\varphi_{31}'}{\varphi_{31}} = -\sqrt{i\alpha R(-c)} + \frac{A_1'}{A_1}$$

$$\frac{\varphi_{32}'}{\varphi_{31}} = \left\{-\sqrt{i\alpha R(-c)}\frac{A_2}{A_1} + \frac{A_2'}{A_1}\right\} e^{-P}, \qquad \frac{\varphi_{41}}{\varphi_{42}} = \frac{B_1}{B_2} e^{-P} \qquad (1.7)$$

$$\frac{\varphi_{41}'}{\varphi_{42}} = \left\{\sqrt{i\alpha R(-c)}\,\frac{B_1}{B_2} + \frac{B_1'}{B_2}\right\} e^{-P}, \qquad \frac{\varphi_{42}'}{\varphi_{42}} = \sqrt{i\alpha R(-c)} + \frac{B_2'}{B_2}$$

To assess the order of magnitude of these ratios we notice that the real part of the integral $\operatorname{Re} P = O(\sqrt{\alpha R})$. Dividing the third column of the determinant (1.3) by ϕ_{31}, and the fourth by ϕ_{42} and assuming αR so large that all the ratios (1.7), apart from ϕ_{31}'/ϕ_{31} and ϕ_{42}'/ϕ_{42}, are very small, we obtain

$$\frac{\varphi_{31}}{\varphi_{31}'}\frac{\varphi_{42}}{\varphi_{42}'} f_4 - \frac{\varphi_{31}}{\varphi_{31}'} f_3 - \frac{\varphi_{42}}{\varphi_{42}'} f_2 + f_1 = 0 \tag{1.8}$$

Here

$$f_1 = \begin{vmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{vmatrix}, \quad f_2 = \begin{vmatrix} \varphi_{11} & \varphi_{12}' \\ \varphi_{21} & \varphi_{22}' \end{vmatrix}, \quad f_3 = \begin{vmatrix} \varphi_{11}' & \varphi_{12} \\ \varphi_{21}' & \varphi_{22} \end{vmatrix}, \quad f_4 = \begin{vmatrix} \varphi_{11}' & \varphi_{12}' \\ \varphi_{21}' & \varphi_{22}' \end{vmatrix} \tag{1.9}$$

These quantities are composed simply of the solutions of the degenerate equation (1.4).

In the calculation of the eigenvalues from Equation (1.8) it is necessary to be more accurate in expressing the ratios ϕ_{31}/ϕ_{31}' and ϕ_{42}/ϕ_{42}' than in Formulas (1.7). The expressions of the functions ϕ_3 and ϕ_4 close to the walls are obtained by expanding the solution of Equation (1.1) as a series $\phi = \chi^{(0)} + \epsilon\,\chi^{(1)} \ldots$ in powers of the small parameter $\epsilon = (\alpha R)^{-1/3}$, and substituting the symbol $\eta = (y - y_c)/\epsilon$, where y_c is the coordinate of the corresponding critical point. The zero-order terms of the expansion are the solutions of Stokes' equation

$$w_c'\eta\chi^{(0)''} + i\chi^{(0)\,\mathrm{IV}} = 0 \tag{1.10}$$

where w_c' is the slope of the velocity profile at the critical point.

In [4,5] it is shown that the asymptotic solutions of the Orr-Sommerfeld equation remain regular in a complex domain of y containing the points y_1 and y_2, if the real part of the integral $Y(y)$ increases monotonically as y varies from y_1 to y_2. Accordingly $\operatorname{Re} Y(y) \gtrless 0$ when $y \gtrless y_{c1}$, and by virtue of (1.5) the solution ϕ_3 must decrease as y approaches y_2, whilst ϕ_4, on the other hand, increases. For approximations to the solutions ϕ_3 and ϕ_4 we therefore take those solutions of Equation (1.10) which vary in the stated manner. These solutions are

$$\varphi_3 \approx \chi_1^{(0)}, \qquad \varphi_4 = \chi_2^{(0)}$$

where

$$\chi_l^{(0)} = \int_0^{\eta} \int_{\theta}^{\eta} \sqrt{\zeta} H_{\frac{1}{3}}^{(1)}\left[\frac{2}{3}(i\alpha_{0l}\zeta)^{3/2}\right] d\zeta d\eta, \qquad \alpha_{0l} = (w_{cl}')^{1/3} \qquad \begin{pmatrix}\theta = (-1)^{l+1}\infty \\ l = 1,\,2\end{pmatrix} \quad (1.11)$$

Here $H_{1/3}^{(1)}$ is the Hankel function of the first kind. Then we obtain

$$\frac{\varphi_{31}}{\varphi_{31}'} \approx \frac{\chi_{11}^{(0)}}{\chi_{11}^{(0)'}}, \qquad \frac{\varphi_{42}}{\varphi_{42}'} \approx \frac{\chi_{22}^{(0)}}{\chi_{22}^{(0)'}}$$

Here

$$\frac{\chi_{ll}^{(0)}}{\chi_{ll}^{(0)'}} = (y_l - y_{cl}) F(z_l), \quad z_l = -\alpha_{0l}\eta_l, \quad \eta_l = \frac{y_l - y_{cl}}{\varepsilon} \quad (l = 1,2) \quad (1.12)$$

$$F(z) = \int_{\infty}^{-z} \int_{\infty}^{\eta} \sqrt{\zeta} H_{1/3}^{(1)}\left[\tfrac{2}{3}(i\zeta)^{3/2}\right] d\zeta d\eta \Big/ \left(-z \int_{\infty}^{-z} \sqrt{\zeta} H_{1/3}^{(1)}\left[\tfrac{2}{3}(i\zeta)^{3/2}\right] d\zeta\right) \quad (1.13)$$

Tables of the Tietjens function $F(z)$ are to be found in [1,6].

The solutions ϕ_1 and ϕ_2 of the degenerate equation are found by expanding them in power series of the wave number α

$$\varphi_1 = (w - c) \sum_{n=0}^{\infty} \alpha^{2n} h_{2n}(y), \qquad h_{2n}(y) = \int_0^y a^{-1} \int_0^y a h_{2n-2}(y)\, dy\, dy, \qquad h_0 = 1 \quad (1.14)$$

$$\varphi_2 = (w - c) \sum_{n=0}^{\infty} \alpha^{2n} k_{2n+1}(y), \qquad k_{2n+1}(y) = \int_0^y a^{-1} \int_0^y a k_{2n-1}(y)\, dy\, dy \quad (1.15)$$

$$a = (w - c)^2, \qquad k_1 = \int_0^y a^{-1}\, dy$$

In these expressions the integration is carried out in the complex plane of y in the neighborhood of the critical points according to the following rule: when $w_c' > 0$ the path of integration falls below, and when $w_c' < 0$ it falls above the critical point (Fig. 2). This follows immediately from the stated condition on the monotonic growth of Re $Y(y)$.

Let us introduce yet other expressions for the derivatives ϕ_1' and ϕ_2':

$$\varphi_1' = w' \sum_{n=0}^{\infty} \alpha^{2n} h_{2n}(y) + \frac{\alpha^2}{w - c} \sum_{n=0}^{\infty} \alpha^{2n} h_{2n+1}(y)$$

$$h_{2n+1}(y) = \int_0^y a \int_0^y a^{-1} h_{2n-1}(y)\, dy\, dy, \qquad h_1 = \int_0^y a\, dy \quad (1.16)$$

$$\varphi_2' = w' \sum_{n=0}^{\infty} \alpha^{2n} k_{2n+1}(y) + \frac{1}{w-c} \sum_{n=0}^{\infty} \alpha^{2n} k_{2n}(y)$$

$$k_{2n}(y) = \int_0^y a \int_0^y a^{-1} k_{2n-2}(y) dy\, dy, \qquad k_0 = 1 \tag{1.17}$$

In what follows we shall introduce the notation ($n = 0, 1, 2, \ldots;$ $l = 1, 2$):

$$h_n(y_l) = H_n^{(l)}, \qquad \sum_{n=0}^{\infty} \alpha^{2n} H_{2n}^{(l)} = H_+^{(l)}, \qquad \sum_{n=0}^{\infty} \alpha^{2n} H_{2n+1}^{(l)} = H_-^{(l)} \tag{1.18}$$

$$k_n(y_l) = K_n^{(l)}, \qquad \sum_{n=0}^{\infty} \alpha^{2n} K_{2n}^{(l)} = K_+^{(l)}, \qquad \sum_{n=0}^{\infty} \alpha^{2n} K_{2n+1}^{(l)} = K_-^{(l)} \tag{1.19}$$

$$L_1 = H_+^{(1)} K_-^{(2)} - H_+^{(2)} K_-^{(1)}, \qquad L_2 = H_+^{(1)} K_+^{(2)} - \alpha^2 H_-^{(2)} K_-^{(1)}$$

$$L_3 = H_+^{(2)} K_+^{(1)} - \alpha^2 H_-^{(1)} K_-^{(2)}, \qquad L_4 = H_-^{(1)} K_+^{(2)} - H_-^{(2)} K_+^{(1)} \tag{1.20}$$

Then we obtain the following expressions for f_k:

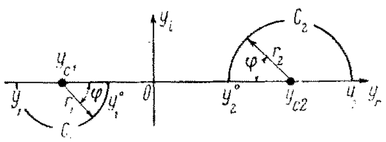

Fig. 2.

$$f_1 = c^2 L_1, \qquad f_2 = -w_2' c L_1 + L_2,$$

$$f_3 = -w_1' c L_1 - L_3 \tag{1.21}$$

$$f_4 = w_1' w_2' L_1 - \frac{w_1'}{c} L_2 + \frac{w_2'}{c} L_3 + \frac{\alpha^2}{c^2} L_4$$

Let us now transform Equation (1.8), setting

$$y_l - y_{cl} = -\frac{c}{w_l'}(1 + \lambda_l) \qquad (l = 1, 2) \tag{1.22}$$

where the quantities λ_l are determined by the given velocity profile and the value of c. (For small c these quantities are generally small in comparison with unity).

Then in place of (1.12) we can write

$$\frac{\chi_{ll}^{(0)}}{\chi_{ll}^{(0)'}} = -\frac{c}{w_l'} F_l \qquad (F_l = (1 + \lambda_l) F(z_l)) \tag{1.23}$$

If we now set $\Phi_l = 1/(1 - F_l)$, then the equation for the eigenvalues (1.8) as a result of the relations (1.21) to (1.23) takes the form

$$c^2 w_1' w_2' L_1 + c w_1' L_2 (\Phi_2 - 1) - c w_2' L_3 (\Phi_1 - 1) + \alpha^2 L_4 (\Phi_1 - 1)(\Phi_2 - 1) = 0 \tag{1.24}$$

2. Let us estimate quantities $H_+^{(l)}$, $H_-^{(l)}$, $K_+^{(l)}$ and $K_-^{(l)}$ appearing in L_k. For this it is necessary first of all to estimate the integrals

$$H_{2n}^{(l)} = \int_0^{y_l} a^{-1} \int_0^y a \int_0^y \ldots \int_0^y a(dy)^{2n}, \qquad H_{2n+1}^{(l)} = \int_0^{y_l} a \int_0^y a^{-1} \int_0^y \ldots \int_0^y a\,(dy)^{2n+1} \tag{2.1}$$

$$K_{2n}^{(l)} = \int_0^{y_l} a \int_0^y a^{-1} \int_0^y \ldots \int_0^y a^{-1}(dy)^{2n}, \qquad K_{2n+1}^{(l)} = \int_0^{y_l} a^{-1} \int_0^y a \int_0^y \ldots \int_0^y a^{-1}(dy)^{2n+1}$$

Let us define the path of integration. We shall skirt the critical points with semi-circles C_l with radii $r_l = |y_l - y_{cl}|$, disposed respectively in the lower and upper half-planes of the complex plane of y (Fig. 2), whilst the remainder of the path is taken along the real axis. Let y_l^0 be the points of intersection of the real axis with the semicircles C_l. For estimates of the integrals it is sufficient to consider one of the integrals $(0, y_1)$. Let the complex variable $y - y(t)$ on the path of integration be a function of the real parameter t, the length of arc along the path of integration, starting from the origin of coordinates. Then

$$t = -y \qquad \text{when } y_1^0 \leqslant y \leqslant 0$$

$$t = -y_1^0 - r_1\varphi \quad \text{when } y - y_{c1} = r_1 e^{i\varphi}, \quad -\pi \leqslant \varphi \leqslant 0 \quad (dy/dt = -ie^{i\varphi})$$

Let us estimate the moduli of the integrals. Evidently

$$|K_{2n}^{(1)}| \leqslant \int_0^{t_1} |a| \int_0^t |a^{-1}| \int_0^t \ldots \int_0^t |a^{-1}|(dt)^{2n} \qquad (n = 1,2,\ldots) \tag{2.2}$$

since

$$\left|\frac{dy}{dt}\right| = 1, \qquad t_1 = -y_1^0 + \pi r_1, \qquad r_1 = \frac{c}{w_1'}(1 + \lambda_1)$$

Moreover

$$\int_0^t |a^{-1}| t^k\,dt \leqslant |a^{-1}| t^{k+1} \tag{2.3}$$

The last relation is obvious when $0 \leqslant t \leqslant \vartheta_1 = -y_1^0$, since in this interval the quantity $|a^{-1}|\,t^k$ is increasing. If, however, $t \geqslant \vartheta_1$, then the following relation holds:

$$\int_0^t |a^{-1}| t^k\,dt = \frac{\vartheta_1^k(1-\varphi)}{w_{c1}'^2 r_1} + O(|\ln r_1|) \tag{2.4}$$

which is obtained by expanding the integrand in the neighborhood of the critical point

$$|a^{-1}| = \frac{1}{w_{c1}'^2 r_1^2}\left|1 - \frac{w_{c1}''}{w_{c1}'} r_1 e^{i\varphi} + \ldots\right| \qquad (t \geqslant \vartheta_1)$$

$$|a^{-1}| = \frac{1}{w_{c1}'^2 (t_{c1} - t)^2}\left[1 - \frac{w_{c1}''}{w_{c1}'}(t_{c1} - t) + \ldots\right] \qquad (t \leqslant \vartheta_1,\ t_{c1} \doteq -y_{c1})$$

From the relation (2.2) by virtue of (2.3) it follows that

$$|K_{2n}^{(1)}| \leqslant \frac{t_1^{2n}}{(2n)!!} \tag{2.5}$$

In just the same way we obtain

$$|K_{2n+1}^{(1)}| \leqslant \int_0^{t_1} |a^{-1}| \frac{t^{2n}}{(2n)!!} dt = \frac{\pi + 1}{w_{c1}'^2 r_1} \frac{\vartheta_1^{2n}}{(2n)!!} + O(|\ln r_1|) \tag{2.6}$$

Since $|a| < 1$ and $\int_0^t |a| dt < 1$, then for $H_{2n+1}^{(1)}$ and $H_{2n}^{(1)}$ we have

$$|H_{2n+1}^{(1)}| \leqslant \frac{t_1^{2n+1}}{(2n+1)!!} \tag{2.7}$$

$$|H_{2n}^{(1)}| \leqslant \int_0^{t_1} |a^{-1}| \frac{t^{2n-1}}{(2n-1)!!} dt = \frac{\pi + 1}{w_{c1}'^2 r_1} \frac{\vartheta_1^{2n-1}}{(2n-1)!!} + O(|\ln r_1|) \tag{2.8}$$

Let us estimate the imaginary parts of the integrals. The expressions in the integrands are complex only on C_1, so that, for example

$$\operatorname{Im} K_{2n}^{(1)} = \operatorname{Im} \int_0^{-\pi} i r_1 a e^{i\varphi} k_{2n-1}\, d\varphi \tag{2.9}$$

where for k_{2n-1} we can write

$$k_{2n-1} = k_{2n-1}(\vartheta_1) + \int_0^{\varphi} i r_1 a^{-1} e^{i\varphi} k_{2n-2}\, d\varphi \tag{2.10}$$

and $k_{2n-1}(\vartheta_1)$ are real quantities, the moduli of which are less than the moduli $|K_{2n-1}|$. According to Formula (2.10), it follows from (2.9) that

$$|\operatorname{Im} K_{2n}^{(1)}| < \sum_{j=1}^{2n} r_1^j |K_{2n-j}^{(1)}| |\operatorname{Im} J_j| \tag{2.11}$$

and likewise

$$|\operatorname{Im} K_{2n+1}^{(1)}| < \sum_{j=1}^{2n} r_1^j |K_{2n-j+1}^{(1)}| \cdot |\operatorname{Im} I_j| \tag{2.12}$$

Similarly

$$|\operatorname{Im} H_{2n}^{(1)}| < \sum_{j=1}^{2n} r_1^{\,j} \, |H_{2n-j}^{(1)}| \cdot |\operatorname{Im} I_j| \tag{2.13}$$

$$|\operatorname{Im} H_{2n+1}^{(1)}| < \sum_{j=1}^{2n} r_1^{\,j} \, |H_{2n-j+1}^{(1)}| \cdot |\operatorname{Im} J_j| \tag{2.14}$$

where

$$J_j = i^j \int_0^{-\pi} a e^{i\varphi} \int_0^{\varphi} a^{-1} e^{i\varphi} \int_0^{\varphi} \ldots \int_0^{\varphi} a^{(-1)^{j+1}} e^{i\varphi} \, (d\varphi)^j \tag{2.15}$$

$$I_j = i^j \int_0^{-\pi} a^{-1} e^{i\varphi} \int_0^{\varphi} a e^{i\varphi} \int_0^{\varphi} \ldots \int_0^{\varphi} a^{(-1)^j} e^{i\varphi} \, (d\varphi)^j \tag{2.16}$$

Expanding the expressions a and a^{-1} in series, it is not difficult to obtain

$$\operatorname{Im} J_1 = 0, \quad \operatorname{Im} I_1 = \frac{\pi}{r_1} \frac{w_{c1}''}{w_{c1}'^3}, \qquad \operatorname{Im} J_2 = \operatorname{Im} I_2 = -\pi r_1 \frac{w_{c1}''}{3w_{c1}'} + O(r_1^2) \tag{2.17}$$

Moreover, it is clear that

$$|J_3| = O(r_1^2), \qquad |J_j| \leqslant O(1), \qquad |I_j| \leqslant O(r_1^{-2}) \qquad (j = 1, 2, \ldots) \tag{2.18}$$

From Formulas (2.17) and the estimates (2.18) we obtain the following estimates of the series (2.11) - (2.14) as regards their main terms:

$$\begin{aligned} |\operatorname{Im} K_{2n}^{(1)}| &= O(r_1^3), \qquad |\operatorname{Im} K_{2n+1}^{(1)}| = O(1) \\ |\operatorname{Im} H_{2n}^{(1)}| &= O(1), \qquad |\operatorname{Im} H_{2n+1}^{(1)}| = O(r_1^3) \end{aligned} \tag{2.19}$$

In what follows we shall need for approximate calculations the formula

$$H_2^{(1)} = K_1^{(1)} H_1^{(1)} + O(\ln r_1) + iO(r_1^3) \tag{2.20}$$

which can be obtained, after estimating the modulus and the imaginary part of the integral

$$H_2^{(1)} - K_1^{(1)} H_1^{(1)} = \int_0^{y_1} a^{-1} \int_{y_1}^{y} a \, dy \, dy$$

by a method similar to that followed above.

The results which we have obtained may be used for the simplification of the equation for the eigenvalues (1.24); we retain in it only the terms of a given order of magnitude. Let us consider the simplest case of all, when the real part of Equation (1.24) is calculated to an accuracy $O(r_1)$, and the imaginary part to $O(r_1^2)$. Moreover, here we shall assume that $\alpha^2 = O(r_1)$, whilst $\Phi = O(1) + iO(r_1)$, as in the case of the symmetrical velocity profile [2].

Then to the given accuracy we set

$$H_+^{(l)} = 1 + \alpha^2 K_1^{(l)} H_{10}^{(l)}, \qquad H_{10}^{(l)} = \int_0^{y_l} w^2\,dy$$

$$H_-^{(l)} = H_{10}^{(l)}, \quad K_+^{(l)} = 1, \quad K_-^{(l)} = K_1^{(l)} \qquad (l = 1, 2) \tag{2.21}$$

Moreover, the real and imaginary parts of the integral $K_1(l)$ are

$$K_{1r}^{(l)} = \frac{1}{w_l' c}, \qquad K_{1i}^{(l)} = \pi \frac{w_{cl}''}{w_{cl}'^3}. \tag{2.22}$$

Using these expressions for the calculation of the quantities L_k, from the real and imaginary parts of Equation (1.24) respectively, we finally obtain

$$\frac{\alpha^2 H_{10}}{c} = \frac{w_1'\Phi_{2r} - w_2'\Phi_{1r}}{\Phi_{1r}\Phi_{2r}}, \qquad \frac{Z_1}{Z_2} = \frac{w_2'}{w_1'}\left(\frac{\Phi_{1r}}{\Phi_{2r}}\right)^2 \tag{2.23}$$

where $H_{10} = H_{10}^{(2)} - H_{10}^{(1)}$, $Z_l = \Phi_{li} + cw_l' K_{1i}(l)$, Φ_{lr} and Φ_{li} are the real and imaginary parts of the functions $\Phi_l (l = 1, 2)$. In the case of the symmetrical velocity profile ($w_2' = -w_1'$, $\Phi_2 = \Phi_1$) these formulas reduce to the formulas presented in [2]

$$\frac{\alpha^2 H_{10}}{2c} = \frac{w_1'}{F_{1r}}, \qquad Z_1 = 0 \tag{2.24}$$

3. As an example of the calculation of the stability of a flow with asymmetric velocity profile in a channel, let us consider the flow (Fig. 1)

$$\begin{aligned} w &= \cos\frac{\pi}{2}\frac{y}{y_1} \ (0 \geqslant y \geqslant y_1) \\ w &= \cos\frac{\pi}{2}\frac{y}{y_2} \ (0 \leqslant y \leqslant y_2) \end{aligned} \qquad (y_1 + y_2 = 1) \tag{3.1}$$

Such a flow is formed by the inflow of fluid into a channel through permeable walls with constant velocities v_{01} and v_{02} ($v_{01}/v_{02} = -y_2/y_1$) under the condition that the quantities $v_{01}H/\nu$ and $v_{02}H/\nu$ are large compared with unity. The arguments of the Tietjens function are in this case

$$z_1 = y_1^{2/3} z, \qquad z_2 = y_2^{2/3} z \qquad \left(z = \left(\frac{2}{\pi}\right)^{2/3} c\sqrt[3]{\alpha R}\right) \tag{3.2}$$

Here for the sake of simplicity it is assumed that the slope of the velocity profile at the critical point is equal to the slope at the corresponding wall of the channel, $w_{cl}' \approx w_l'$ (i.e. $\lambda_l = 0$, or $c^2 \ll 1$).

From Equations (2.23) it follows that

$$\alpha^2 = \pi c \frac{y_1\Phi_{1r} - y_2\Phi_{2r}}{y_1\Phi_{1r}y_2\Phi_{2r}}$$

$$\pi c^2 = \frac{y_1\Phi_{1r}{}^2\Phi_{2i} - y_2\Phi_{2r}{}^2\Phi_{1i}}{y_1F_{1r}{}^2 - y_2F_{2r}{}^2} \tag{3.3}$$

The system of equations (3.2) - (3.3) determines the neutral curve in the plane of the parameters α, R. Great practical interest attaches to the calculation of the critical Reynolds number R_*, for which the first onset of instability occurs in the flow. An estimate of the quantity R_* can be obtained without constructing the whole of the neutral curve, provided that in accordance with (3.2) it is assumed approximately that R attains a minimum when the quantity c, considered as a function of z, is a maximum.

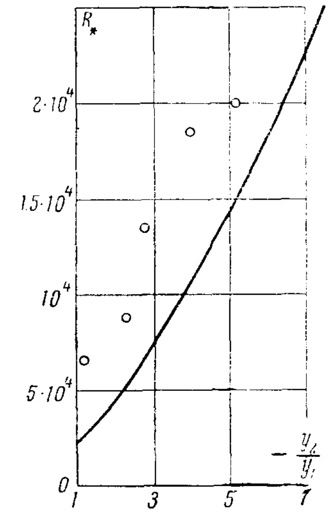

Fig. 3.

For the symmetric profile ($-y_1 = y_2 = 1/2$) such a calculation gives $R_* = 2150$. If, however, $y_1 = -0.4$, then $R_* = 2900$.

Finally, let us consider the limiting case $y_1 \to 0$. As follows from the properties of the Tietjens function, $\Phi_i(z)$ and $\Phi_r(z)$ tend to zero as $z \to 0$. Then as $y_1 \to 0$ we have, for $c = c_{max}$

$$\pi c^2 = \Phi_{1i} = 0.58, \qquad z_1 = 3.21, \qquad \alpha^2 = \frac{-\pi c}{y_1y_2\Phi_{1r}}, \qquad \Phi_{1r} = 1.50 \tag{3.4}$$

and

$$R_* = 1075\sqrt{-\frac{y_2}{y_1}}\left(1 - \frac{y_2}{y_1}\right) \tag{3.5}$$

For example, $R_* = 27400$ when $y_1 = -1/9$, i.e. the asymmetric flow of the type considered is strongly stabilized.

The conclusion as to the stabilization of the flow with increase of the ratio $-y_2/y_1$ ($-y_2/y_1 > 1$) has been verified by the author experimentally. For this purpose a flow was produced in a two-dimensional-channel model, with a velocity profile conforming to the relation (3.1). Two opposite walls of the channel were made out of net, whilst the other two walls and the base of the channel were impermeable. The ratio of the velocity of inflow of air through the walls was changed by changing the resistance of the nets. To minimize external disturbances the air was exhausted from the channel through a critical section. The beginning of the transition to the turbulent regime was observed by the onset of oscillations in the stream. For the measured velocity profile, the values of R_* and y_2/y_1 were determined. The values of the Reynolds number so obtained are shown in Fig. 3. Here is also displayed the theoretical curve $R_*(y_2/y_1)$ computed from Formula (3.5) (which for values of y_1 close to $-1/2$ satisfactorily approximates the value of R_* calculated from Formulas (3.33) without the limiting transition $y_1 \to 0$). The results of the measurements confirm the conclusions obtained from theory.

4. The method described above can be used without fundamental changes to solve the problem of the stability of axial flows in annular channels, if it is accepted that the magnitude of the internal radius of the annular channel is not small compared with the width of the channel.

The differential equation for the amplitude of the perturbation of axisymmetric type in this case has the form

$$(P^2-\alpha^2)^2\varphi = i\alpha R\,[(w-c)(P^2-\alpha^2)-P^2w]\,\varphi \qquad \left(P^2=\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right) \quad (4.1)$$

$$\varphi=\varphi'=0 \text{ when } r=r_1 \text{ and } r=r_2 \qquad (w(r_1)=w(r_2)=0)$$

The solution of the degenerate equation in this case has the same form as for the plane problem, except for the fact that in all the integrals the quantity a must be

$$a=\frac{(w-c)^2}{r} \quad (4.2)$$

and the lower limit of integration is r_* (r_* is the point corresponding to the maximum velocity in the channel). The integration is carried out in the complex plane of r, skirting the critical points below or above according to the sign of w_c' (> 0 or < 0). The solution of the equation for ϕ_3 and ϕ_4 is just the same as in the plane case (y being everywhere replaced by r).

The approximate relations of type (2.23) in this case have the form

$$\frac{\alpha^2 H_{10}}{c}=\frac{w_1' r_2\Phi_{2r}-w_2' r_1\Phi_{1r}}{r_1\Phi_{1r} r_2\Phi_{2r}}, \qquad \frac{Z_1}{Z_2}=\frac{w_2'}{w_1'}\left(\frac{\Phi_{1r}}{\Phi_{2r}}\right)^2\frac{r_1}{r_2} \quad (4.3)$$

Here

$$H_{10}=\int_{r_1}^{r_2}\frac{w^2}{r}dr, \quad Z_l=\Phi_{li}+cw_l'\frac{K_{1i}^{(l)}}{r_l}, \qquad K_{1i}^{(l)}=\pi\frac{r_{cl}^2}{w_{cl}'^3}\left(\frac{w'}{r}\right)'_{r=r_{cl}} \; (l=1,2) \quad (4.4)$$

BIBLIOGRAPHY

1. Lin, C.C., On the stability of two-dimensional parallel flows. *Quart. Appl. Math.* Vol. 3, 1945-46.

2. Lin, C.C., *Teoriia gidrodinamicheskoi ustoichivosti* (*Theory of Hydrodynamic Stability*). IIL, Moscow, 1956.

3. Yuan, S.W., Further investigation of laminar flow in channels with porous walls. *J. Appl. Phys.* Vol. 27, No. 3, 1956.

4. Wasow, W., The complex asymptotic theory of a fourth-order differential equation of hydrodynamics. *Ann. Math.* 49, 1948.

5. Foot, J.R. and Lin, C.C., Some recent investigations in the theory of hydrodynamic stability. *Quart. Appl. Math.* Vol. 8, 1950.

6. Holstein, H., Über die äussere und innere Reibungsschicht bei Störungen laminarer Strömungen. *Z. angew. Math. und Mech.* Vol. 30, Nos. 1-2, 1950.

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